Optimal bundle and minimum cost demonstration

Suppose that the preference relation $\succeq$ is locally non-satiated. Let x^* be a feasible allocation and p a price vector. Prove that the following two conditions are equivalent:

(a) If $y \succeq x^*$ then $p \cdot y \geq p \cdot x^*$.

(b) x^* is a solution to the problem

$$\min p \cdot x \quad \text{s.t.} \quad x \succeq x^*.$$

Solution

Intuition:

Condition (a): This condition states that if an allocation y is at least as good as x^* in terms of preferences, then the cost of y (at prices p) must be at least as high as the cost of x^*. This implies that x^* is cost-effective or minimized given the prices p and the preferences.
Condition (b): This condition states that x^* is the allocation that minimizes the cost (at prices p) among all allocations that are at least as good as x^* in terms of preferences. This means x^* is not only feasible and preferred but also the cheapest option among those preferred.

Both conditions essentially ensure that x^* is a preferred and cost-minimized allocation, making it an optimal choice given the prices and preferences.

Let's see that (a) implies (b). To do this, observe that if x^* is not a solution to the problem

$$\min p \cdot x \quad \text{s.t.} \quad x \succeq x^*,$$

then we can find an allocation $y \succeq x^*$ such that $p \cdot y < p \cdot x^*$. But this contradicts (a). Therefore, (a) implies (b).

Now, let's see that (b) implies (a). Let $y \succeq x^*$. Since x^* is a solution to the problem

$$\min p \cdot x \quad \text{s.t.} \quad x \succeq x^*,$$

then $p \cdot x^* \leq p \cdot y$.

In conclusion, we have demonstrated that conditions (a) and (b) are equivalent. Condition (a) ensures that any allocation y that is at least as good as x^* must have a cost at least as high as x^*. Conversely, condition (b) confirms that x^* is the minimum cost allocation among all allocations that are at least as good as x^*. Together, these conditions guarantee that x^* is an optimal allocation, balancing both preference satisfaction and cost efficiency.